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THE STAR PUZZLE : COMPUTATIONAL ASPECT

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Abstract

The *star puzzle* is a variant of the Tower of Hanoi problem, where, in addition to the usual three pegs, S, P and D, there is a fourth one such that all disc movements are either to or from the fourth peg. Letting MS(n) be the minimum number of moves required to solve the star puzzle, MS(n) satisfies the recurrence relation below

$$MS(n) = \min_{1 \le k \le n-1} \left\{ 2MS(n-k) + 3^k - 1 \right\}.$$

This paper studies the computational aspect of the star puzzle.

Keywords : Star puzzle, three-in-a-row puzzle, recurrence relation, algorithm

1. Introduction

The *star puzzle*, posed and solved by Stockmeyer [1], is as follows : There are three pegs, S, P and D, arranged in an equilateral triangle, and there is the fourth peg at the center 0. Each disc movement must be either to or from 0, that is, direct moves of discs between any two of the pegs S, P and D are not allowed. Initially, the n discs of different sizes, designated as $D_1, D_2, \ldots, D_n$, are placed on the *source peg*, S, in a tower (in small-on-large ordering, with the largest disc, D_n, at the bottom, the second largest disc, D_{n-1}, above it, and so on, with the smallest disc, D_1, at the top). The problem is to shift this tower of n discs from the peg S to the *destination peg*, D, in minimum number of moves, using the *auxiliary peg* P, under the condition that each move can transfer only the topmost disc from one peg to another such that no disc is ever placed on top of a smaller one. The situation is depicted below.

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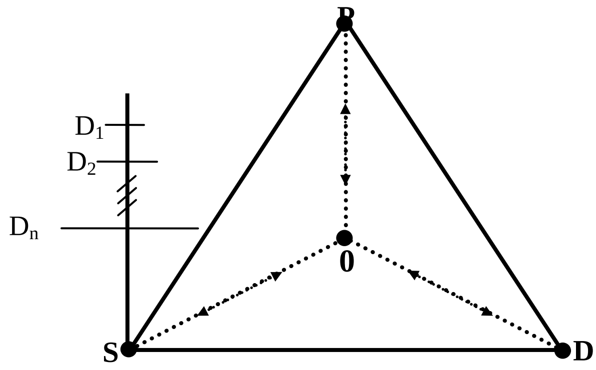

Figure 1.1 : The Star Puzzle

Let MS(n) denote the minimum number of moves required to solve the above problem. Then, MS(n) satisfies the following recurrence equation, due to Stockmeyer [1].

$$MS(n) = \min_{1 \le k \le n-1} \left\{ 2MS(n-k) + 3^k - 1 \right\} \quad n \ge 2, \qquad (1.1)$$

with

$$MS(0) = 0, MS(1) = 2. \qquad (1.2)$$

Recall that, to find (1.1), the scheme below is followed :

Step 1: Move the tower of the topmost n – k discs from the peg S to the peg P, using all the four pegs available, in (minimum) MS(n – k) number of moves.

Step 2: Transfer the remaining k discs from the peg S to the peg D, using the three pegs available. This corresponds to the *three-in-a-row puzzle*, due to Scorer, Grundy and Smith [2], and the (minimum) number of moves required is $3^k - 1$.

Step 3 : Finally, move the tower from the peg P on top of the discs on the peg D, again in (minimum) MS(n – k) number of moves.

The total number of moves required is

$$FS(n, k) \equiv 2MS(n-k) + 3^k - 1, \qquad (1.3)$$

where k ($1 \le k \le n-1$) is to be determined so as to minimize FS(n, k).

Note that, with only one disc, the transfer is made first from the peg S to the auxiliary peg P, next the disc is moved from the peg P to the destination peg D. The number of moves involved is thus 2.

The following results have been established by Majumdar [3].

***Lemma 1.1* :** MS(n) is an even (positive) integer for any integer $n \geq 1$.

***Lemma 1.2* :** For $n \geq 4$, MS(n) is not attained at $k = n - 1$.

***Lemma 1.3* :** MS(n +1) > MS(n), $n \geq 1$.

***Lemma 1.4* :** For any $n \geq 1$, $MS(n + 2) - MS(n + 1) \leq 2\{MS(n + 1) - MS(n)\}$.

The problem was later taken up by Majumdar [4], who derived some local-value relationships satisfied by the optimal value function MS(n). Stockmeyer [1] gave a sketch of the proof that MS(n) is attained at the unique point $k = \left\lfloor \frac{\ln(b_n)}{\ln 3} \right\rfloor + 1$, with

$$MS(n) = \sum_{m=1}^{n} a_m = 2\sum_{m=1}^{n} b_m, \tag{1.1}$$

where $\{b_n\}_{n=1}^{\infty}$ is the sequence of numbers, arranged in (strictly increasing order), defined as follows :

$$b_n = 2^i 3^m;\ i \geq 0,\ m \geq 0.$$

However, the argument given by Stockmeyer [1] to derive (1.1) is rather heuristic in nature, and is not supported by any theoretical development. Moreover, to find MS(n), we have to keep track of the sequence of numbers $\{b_n\}_{n=1}^{\infty}$ about which much is not known. In this paper, we give an algorithm which calculates MS(n) recursively in n. The proposed algorithm also finds the point k at which MS(n) is attained. This is given in Section 3. In the next Section 2, we give some preliminary results.

2. Some Preliminary Results

The following results have been derived by Majumdar [4].

***Lemma 2.1* :** For any integer $n \geq 1$,

(a) $MS(n + 2) - MS(n + 1) > MS(n + 1) - MS(n)$,

(b) MS(n) is attained at a unique value of k.

***Corollary 2.1* :** If, for some integer $n \geq 1$, MS(n) is attained at the point $k = k_1$ and $MS(n + 1)$ is attained at $k = k_2$, then $k_1 \leq k_2 \leq k_1 + 1$.

***Corollary 2.2* :** If, for some integer $n \geq 1$, MS(n) is attained at the point $k = K$ and $MS(n+1)$ is attained at $k = K+1$, then $MS(n+2)$ must be attained at $k = K+1$.

***Lemma 2.2* :** Let, for some integer $n \geq 2$,

$$MS(n) - MS(n-1) = 2^s \text{ for some integer } s \geq 1. \qquad (2.1)$$

Then, $MS(n-1)$ and MS(n) both are attained at the same value of k.

***Lemma 2.3* :** Let, for some integer $n \geq 1$,

$$MS(n) - MS(n-1) = 2.3^{\ell} \text{ for some integer } \ell \geq 0. \qquad (2.2)$$

Let MS(n) be attained at $k = K$. Then, $MS(n-1)$ is attained at $k = K-1$, and $MS(n+1)$ is attained at $k = K$. Moreover, $MS(n-1)$ and MS(n) satisfy the relationship (2.2) (for some integer $n \geq 1$) if and only if $MS(n-1)$ and MS(n) are attained at different (consecutive) values of k.

***Lemma 2.4* :** Let $N \geq 1$ be such that $MS(N-1)$ is attained at $k = K-1$ and MS(N) is attained at $k = K$, so that

$$MS(N) - MS(N-1) = 2.3^{K-1}. \qquad (2.4)$$

Then, there is an integer $M \geq 1$ such that

$$MS(N+M+1) - MS(N+M) = 2.3^{K}. \qquad (2.5)$$

We now prove the following result.

***Lemma 2.5* :** Given any integer $K \geq 1$, there is an integer $N \geq 1$ such that MS(N) is attained at the point $k = K \geq 1$.

Proof. The proof is by induction on K. The result is true for $K = 1$ with $N = 1$. So, we assume that the result is true for some integer $K \geq 1$, that is, we assume that, for K (≥ 1), there is an integer N such that MS(N) is attained at $k = K$, so that

$$MS(N) = 2MS(N-K) + 3^K - 1.$$

Now, by Corollary 2.1, $MS(N+1)$ is attained either at $k = K$, or else, at $k = K+1$. In the latter case, the proof by induction is complete. Otherwise, $MS(N+1)$ is attained at $k = K$, so that

$$MS(N+1) = 2MS(N-K+1) + 3^K - 1 < 2MS(N-K) + 3^{K+1} - 1,$$

and hence

$$MS(N+1) - MS(N) < 3^K.$$

Now, if $MS(N+2)$ is attained at $k=K+1$, the proof is complete; otherwise

$$MS(N+2)=2MS(N-K+2)+3^K-1<2MS(N-K+1)+3^{K+1}-1,$$

giving

$$MS(N+2)-MS(N+1)<3^K.$$

Continuing in this way, in the worst case, we get the sequence of functions (for some integer m), $MS(N+1)$, $MS(N+2)$, …, $MS(N+m)$, …, each of which is attained at the point $k=K$, with

$$MS(N+i)-MS(N+i-1)<3^k,\ i=1,2,\ldots\ .$$

But, by part (a) of Lemma 2.1, the sequence $\{MS(N+i)-MS(N+i-1)\}_{i=1}^{\infty}$ is strictly increasing in i (≥ 1), and hence, there is an integer m (≥ 1) such that

$$MS(N+m)-MS(N+m-1)\geq 3^K.$$

For the minimum such m, $m=M$, say, $MS(N+M-1)$ is attained at the point $k=K$ but $MS(N+M)$ is attained at $k=K+1$, with

$$MS(N+M)-MS(N+M-1)=3^K.$$

Thus, corresponding to the integer $K+1$, we find an integer, namely, $N+M$, such that the function $MSM(N+M)$ is attained at $k=K+1$.

Let the sequence of numbers $\{a_n\}_{n=1}^{\infty}$ be defined by

$$a_n=MS(n)-MS(n-1),\ n\geq 1. \tag{2.5}$$

Let $m_j\geq 1$ be the integer, defined as follows :

$$a_{m_j}=MS(m_j)-MS(m_j-1)=2.3^j;\ j\geq 0,$$

with

$$m_0=1,\ m_1=3.$$

Then, $MS(m_j-1)$ is attained at $k=j$, and for all n with $m_j\leq n\leq m_{j+1}-1$, $MS(n)$ is attained at $k=j+1$.

Let the integers $k_j\geq 1$ be defined as follows :

$$a_{k_j}=MS(k_j)-MS(k_j-1)=2.2^j;\ j\geq 0,$$

with

$$k_0=1,\ k_1=2.$$

The result below is due to Majumdar [4].

***Lemma 2.6* :** For all $j \geq 1$, $MS(k_j)$ is attained at $k = k_j - k_{j-1}$.

Let the sequence of numbers $\{b_n\}_{n=1}^{\infty}$, arranged in (strictly) increasing order, be defined as follows :

$$b_n = 2^i 3^m;\ i \geq 0,\ m \geq 0. \qquad (2.6)$$

The first few terms of the sequence $\{b_n\}_{n=1}^{\infty}$ are

$$1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 214, \ldots . \qquad (2.7)$$

Lemma 2.7, due to Matsuura [5], gives a recurrence relation satisfied by $\{b_n\}_{n=1}^{\infty}$.

***Lemma 2.7* :** Let n be such that $2^i < b_n < 2^{i+1}$ for some integer $i \geq 1$. Then,

$$b_n = 3b_{n-i-1}.$$

Note that, in order to use the recurrence relation given in Lemma 2.7 above, we have to find j such that $2^i < b_n < 2^{i+1}$. However, in the current literature, this is not available, and remains an open problem.

The solution of the recurrence relation (1.1), proposed by Stockmeyer [1], is given below.

***Theorem 2.1* :** For $n \geq 1$, MS(n) is attained at the (unique) point $k = \left\lfloor \frac{\ln(b_n)}{\ln 3} \right\rfloor + 1$, with

$$MS(n) = \sum_{m=1}^{n} a_m = 2\sum_{m=1}^{n} b_m.$$

As has already been mentioned, the argument given by Stockmeyer [1] in proving Theorem 2.1 is heuristic. Moreover, since both MS(n) and the point k at which MS(n) is attained involve the sequence of numbers $\{b_n\}_{n=1}^{\infty}$, from the point of view of application, Theorem 2.1 is of no use to find MS(n) nor the point k at which MS(n) is attained. For small values of n, MS(n) may be calculated readily, using Theorem 2.1 as well as the values listed in (2.7). For example, from (2.7), we see that $b_{10} = 18$, so that, by Theorem 2.1, MS(10) is attained at $k = 3$ with $MS(10) = 158$. But for large n, Theorem 2.1 is not applicable, since the recurrence relation given in Lemma 2.7 can not be applied. To circumvent this drawback, we give a recursive algorithm in the next Section 3, which calculates MS(n) recursively in n.

3. Computational Aspect

From Corollary 2.1, we see that, if MS(n) is attained at the point $k = k_1$ and MS(n + 1) is attained at the point $k = k_2$, then, $k_1 \leq k_2 \leq k_1 + 1$. This result enables us to calculate recursively the value(s) of k at which MS(n) is attained, as well as the values of MS(n). Thus, if MS(n – 1) is attained at k = K, then MS(n) is attained either at k = K, or else, at k = K + 1, so that the problem of finding MS(n) and the value of k at which MS(n) is attained reduces to the problem below

$$MS(n) = \min\{2MS(N-K) + 3^K - 1,\ 2MS(N-K-1) + 3^{K+1} - 1\}.$$

To start with, we note that, MS(1) = 2, and MS(2) is attained at the unique point k = 1, with $MS(2) \equiv 6$ and k(2) = 1. We have the following result.

Lemma 3.1 **:** For $n \geq 3$, MS(n) is attained at $k \geq 2$.

***Proof*.** We first consider the function

$$FS(3, k) = 2MS(3-k) + 3^k - 1,\ 1 \leq k \leq 3.$$

Since

$$FS(3, 1) = 2MS(2) + 2 = 14 > FS(3,2) = 2MS(1) + 8 = 12,$$

it follows that MS(3) is attained at a point $k \geq 2$. The result now follows for all $n \geq 3$, by virtue of Corollary 2.1.

The algorithm to find the value of k at which MS(n) is attained as well as the value of MS(n) is given below.

Algorithm 3.1 **:** Algorithm to find MS(n) and the point k at which MS(n) is attained

/ NN pre-determined integer /

For n = 1, 2, ..., NN

$S(n) = 3^n - 1$

/ Initialization /

MS(1) = 2

MS(2) = 6

k(2) = 1

/ Determination of MS(n) and k(n) /

For $n = 3, 4, ..., NN$

$k = k(n-1)$

$T1 = 2MS(n-k) + S(k)$

$T2 = 2MS(n-k-1) + S(k+1)$

If $T1 < T2$ then

$MS(n) = T1$

$k(n) = k$

else

$MS(n) = T2$

$k(n) = k + 1$

In Algorithm 3.1 above, the quantities T1 and T2 are compared to find MS(n) and then is determined the point k at which MS(n) is attained. For example, to find MS(3) (corresponding to $n = 3$) and the point k at which MS(3) is attained, the algorithm sets

$$k = k(2) = 1,$$

and then calculates T1 and T2 as follows :

$$T1 = 2MS(2) + S(1) = 14,$$

$$T2 = 2MS(1) + S(2) = 12.$$

Since $T2 < T1$, it follows that $MS(3) = 12$ with $k(3) = 2$.

Algorithm 3.1 calculates MS(n) recursively for $3 \le n \le NN$. To do so, we need the values of S(n) for $1 \le n \le NN$ for the calculation of T1 and T2. Note that S(n) grows very rapidly; however, it is sufficient to calculate S(n) for $1 \le n \le NN/2$.

Now, note that

$$T1 - T2 = 2[MS(n-k) - MS(n-K-1)] - 2.3^k.$$

Thus,

$$T1 < T2 \text{ if and only if } MS(n-k) - MS(n-k-1) < 3^k.$$

This observation leads to a second recursive algorithm, given below.

***Algorithm 3.2* :** Algorithm to find MS(n) and the point k at which MS(n) is attained

/ NN pre-determined integer /

For n = 1, 2, ..., NN

$S(n) = 3^n - 1$

/ Initialization /

$MS(1) = 2$

$MS(2) = 6$

$k(2) = 1$

/ Determination of MS(n) and k(n) /

For n = 3, 4, ..., NN

$k = k(n - 1)$

$M = MS(n - k) - MS(n - k - 1)$

If $M < 3^k$ then

$k(n) = k$

$MS(n) = 2MS(n - k(n)) + S(k(n))$

else

$k(n) = k + 1$

$MS(n) = 2MS(n - k(n)) + S(k(n))$

In Algorithm 3.1 and Algorithm 3.2, we start with the following expression :

$$MS(n-1) = 2MS(n-k-1) + 3^k - 1,$$

and then proceed to find MS(n). Note that, by assumption, $MS(n-1)$ is attained at k. Now, if MS(n) is attained at $k + 1$, then

$$MS(n) = 2MS(n-k-1) + 3^{k+1} - 1,$$

so that

$$MS(n) - MS(n-1) = 2.3^k,$$

giving

$$MS(n) = MS(n-1) + 2.3^k. \qquad (*)$$

On the other hand, if MS(n) is attained at k, then

$$MS(n) = 2MS(n-k) + 3^k - 1,$$

which gives

$$MS(n) - MS(n-1) = 2[MS(n-k) - MS(n-k-1)].$$

Then,

$$MS(n) = MS(n-1) + 2[MS(n-k) - MS(n-k-1)]. \qquad (**)$$

Using the expressions of MS(n), given in (*) and (**) respectively, the calculations of MS(n) in Algorithm 3.2 may be simplified. Thus, for example, starting with the fact that MS(2) is attained at $k = 1$ with $MS(2) = 6$, for $n = 3$, we see that

$$M = MS(3-1) - MS(3-1-1) = MS(2) - MS(1) = 4 > 3.$$

Hence, MS(3) is attained at $k = 2$, with

$$MS(3) = MS(2) + 2 \times 3 = 12.$$

Incorporating these facts in Algorithm 3.2, we get the following version.

Algorithm 3.3 **:** Algorithm to find MS(n) and the point k at which MS(n) is attained

/ NN pre-determined integer /

/ Initialization /

$MS(1) = 2$

$MS(2) = 6$

$k(2) = 1$

/ Determination of MS(n) and k(n) /

For $n = 3, 4, ..., NN$

$k = k(n-1)$

$M = MS(n-k) - MS(n-k-1)$

If $M < 3^k$ then

$$k(n) = k$$

$$MS(n) = MS(n-1) + 2M$$

else

$$k(n) = k + 1$$

$$MS(n) = MS(n-1) + 2.3^k$$

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