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GENERAL DIVIDED DIFFERENCE POLYNOMIAL INTERPOLATION FORMULA: A NEW APPROACH

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Abstract

In this paper, a simple introduction about the interpolation method and the new approach of (forward) divided differences are given when the arguments are unevenly spaced. These divided differences are carried out and applied to develop a new polynomial interpolation formula. In accordance with idea of these new (forward) divided differences, the leading differences can be deduced when the arguments are evenly spaced. In the development of the new divided difference table, the advantages are showed according to the speedup of human labour and efficiency. Finally, to implement the new approach via computer an algorithm and FORTRAN77 program are carried out to construct the new divided differences for a given data points and to estimate the value of a function at any tabulated or non-tabulated points.

1. Introduction

In mathematics, the interpolation is the determination or estimation of the value of $y = f(x)$, or a function of x, from a given set of certain known values of the function. If $x_0 < x_1 < \ldots < x_n$ and the corresponding values $y_0 = f_0 = f(x_0)$, $y_1 = f_1 = f(x_1), \ldots, y_n = f_n = f(x_n)$ are known, and if $x_0 < x < x_n$, then the process of estimating the value of $f(x)$ is said to be interpolation. On the other hand, if $x < x_0$ or $x > x_n$, then the process of estimating the value of $f(x)$ is said to be extrapolation. If $x_0,\ x_1,\ x_2,\ x_3, \ldots, x_n$ are given along with the corresponding values $f_0, f_1, f_2, \ldots, f_n$, the interpolation may be regarded as the estimation of a function $y = f(x)$ whose graph passes through all the $(n+1)$ points $(x_k, y_k = f_k)$ for $k = 0, 1, 2, \ldots, n$. There may have many such functions, but the simplest one is the polynomial interpolation function, like $f(x) = \varphi(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$ of degree n with the constants a_i's such that $\varphi(x_i) = f_i$ for $i = 0, 1, 2, \ldots, n$. There is exactly one such interpolating polynomial of degree n or less than n, that is, of degree at most n.

If the x_i's are equally spaced, say by some factor h, then the following formula of Sir Isaac Newton produces a polynomial function that fits the given $(n+1)$ data points:

$$f(x) = a_0 + a_1 \frac{(x - x_0)}{h} + a_2 \frac{(x - x_0)(x - x_1)}{2!\, h^2} + \cdots + a_n \frac{(x - x_0)(x - x_1)\ldots(x - x_n)}{n! h^n} \qquad (1a)$$

If the x_i's are not equally spaced, then the following formula of Sir Isaac Newton produces a polynomial function that fits the given (n+1) data points:

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1)\ldots(x - x_n) \qquad (1b)$$

where $a_0 = f_0$ is the leading term; and

$$a_1 = f[x_0,\ x_1],\ a_2 = f[x_0,\ x_1,\ x_2], \ldots, a_n = f[x_0,\ x_1,\ \ldots x_n]$$

are the leading divided differences, and are estimated in a clumsy way.

In this paper, I have formulated a (forward) divided difference polynomial interpolation formula with (forward) divided differences which are estimated in a more efficient and sophisticated way.

2. (Forward) Divided Differences: A new approach

Let $f_0,\ f_1,\ f_2,\ f_3, \ldots\ldots, f_n$ be a set of $n+1$ values of the function $y = f(x)$ corresponding to the values $x_0,\ x_1,\ x_2,\ x_3, \ldots, x_n$ of the independent variable x. Suppose that $x_0,\ x_1,\ x_2,\ x_3, \ldots, x_n$ are not necessarily equally spaced.

Then the first order divided differences are denoted by $\Delta^1_D f_k$, ($k = 1, 2, 3, 4, \ldots\ldots, n$) and are given by

$$\Delta^1_D f_k = \frac{f_k - f_0}{x_k - x_0},\ k = 1, 2, 3, 4, \ldots\ldots, n\ .$$

The second order divided differences are denoted by $\Delta^2_D f_k$, ($k = 2, 3, 4, \ldots\ldots, n$) and are defined as:

$$\Delta^2_D f_k = \frac{\Delta^1_D f_k - \Delta^1_D f_1}{x_k - x_1},\ k = 2, 3, 4, 5, \ldots\ldots, n$$

The third order divided differences are denoted by $\Delta^3_D f_k$, ($k = 3, 4, 5, 6, \ldots\ldots, n$) and is defined as:

$$\Delta^3_D f_k = \frac{\Delta^2_D f_k - \Delta^2_D f_2}{x_k - x_2},\ k = 3, 4, 5, 6, \ldots\ldots, n$$

Similarly, the n^{th} divided difference of $f(x)$ is denoted by $\Delta^n_D f_n$ and is given by:

$$\Delta^n_D f_n = \frac{\Delta^{n-1}_D f_n - \Delta^{n-1}_D f_{n-1}}{x_n - x_{n-1}}\ .$$

3. *Construction of Difference Table for FIVE arguments*

Let $f_0, f_1, f_2, f_3, \ldots\ldots, f_n$ be a set of $n+1=5$ values of the function $y = f(x)$ corresponding to the values $x_0, x_1, x_2, x_3, \ldots, x_n$ of the independent variable x. Then we can construct a difference as follows:

x	$f(x)$	1st order Difference	2nd order Difference	3ird order Difference	4th order Difference
x_0	f_0				
x_1	f_1	$\Delta_D^1 f_1 = \frac{f_1 - f_0}{x_1 - x_0}$			
x_2	f_2	$\Delta_D^1 f_2 = \frac{f_2 - f_0}{x_2 - x_0}$	$\Delta_D^2 f_2 = \frac{\Delta_D^1 f_2 - \Delta_D^1 f_1}{x_2 - x_1}$		
x_3	f_3	$\Delta_D^1 f_3 = \frac{f_3 - f_0}{x_3 - x_0}$	$\Delta_D^2 f_3 = \frac{\Delta_D^1 f_3 - \Delta_D^1 f_1}{x_3 - x_1}$	$\Delta_D^3 f_3 = \frac{\Delta_D^2 f_3 - \Delta_D^2 f_2}{x_3 - x_2}$	
x_4	f_4	$\Delta_D^1 f_4 = \frac{f_4 - f_0}{x_4 - x_0}$	$\Delta_D^2 f_4 = \frac{\Delta_D^1 f_4 - \Delta_D^1 f_1}{x_4 - x_1}$	$\Delta_D^3 f_4 = \frac{\Delta_D^2 f_4 - \Delta_D^2 f_2}{x_4 - x_2}$	$\Delta_D^4 f_4 = \frac{\Delta_D^3 f_4 - \Delta_D^3 f_3}{x_4 - x_3}$

***Table*-1:** *Difference Table*

4. *Development of General polynomial interpolation formula*

Let $y = f(x)$ be a function which takes the set of n values $f_0, f_1, f_2, f_3, \ldots\ldots, f_n$ corresponding to the values $x_0, x_1, x_2, x_3, \ldots, x_n$ of the independent variable x.

We are required to establish a polynomial $\phi(x) = P_n(x)$ of degree less than or equal to n, such that $f(x)$ and $\phi(x)$ take the same values at the tabulated points $x_0, x_1, x_2, x_3, \ldots, x_n$.

$$\therefore\ \phi(x_0) = f_0,\ \phi(x_1) = f_1,\ \phi(x_2) = f_2,\ \phi(x_3) = f_3, \ldots\ldots, \phi(x_n) = f_n \tag{2}$$

Let us consider $\phi(x) = P_n(x)$ may be written as follows:

$$\phi(x) = P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) \\ + \cdots\cdots + a_n(x - x_0)(x - x_1)(x - x_2)\ldots\ldots(x - x_{n-1}) \tag{3}$$

Putting $x = x_0,\ x = x_1,\ x = x_2,\ x = x_3, \ldots, x = x_n$ successively in equation (3) and at the same time using equation (2), we can find the coefficients $a_0, a_1, a_2, a_3, \ldots, a_n$.

If $x = x_0$, then from equation (3), we have

$$\phi(x_0) = f_0 = a_0 \Rightarrow a_0 = f_0$$

If $x = x_1$, then from equation (3), we have

$$\phi(x_1) = f_1 = a_0 + a_1(x_1 - x_0)$$
$$\Rightarrow f_1 = f_0 + a_1(x_1 - x_0) \quad \Rightarrow a_1(x_1 - x_0) = f_1 - f_0 \quad \therefore \quad a_1 = \frac{f_1 - f_0}{x_1 - x_0} = \Delta_D^1 f_1$$

If $x = x_2$, then from equation (3), we have

$$\phi(x_2) = f_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$
$$\Rightarrow a_2(x_2 - x_0)(x_2 - x_1) = f_2 - a_0 - (x_2 - x_0)a_1$$
$$\Rightarrow a_2(x_2 - x_0)(x_2 - x_1) = f_2 - f_0 - (x_2 - x_0)\Delta_D^1 f_1$$
$$\Rightarrow a_2(x_2 - x_1) = \frac{f_2 - f_0}{x_2 - x_0} - \Delta_D^1 f_1 \quad \Rightarrow a_2(x_2 - x_1) = \Delta_D^1 f_2 - \Delta_D^1 f_1$$
$$\therefore \quad a_2 = \frac{\Delta_D^1 f_2 - \Delta_D^1 f_1}{x_2 - x_1} = \Delta_D^2 f_2$$

If $x = x_3$, then from equation (3), we have

$$\phi(x_3) = f_3 = a_0 + a_1(x_3 - x_0) + a_2(x_3 - x_0)(x_3 - x_1) + a_3(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)$$

$$\Rightarrow a_3(x_3 - x_0)(x_3 - x_1)(x_3 - x_2) = f_3 - a_0 - (x_3 - x_0)a_1 - (x_3 - x_0)(x_3 - x_1)a_2$$

$$\Rightarrow a_3(x_3 - x_0)(x_3 - x_1)(x_3 - x_2) = f_3 - f_0 - (x_3 - x_0)\Delta_D^1 f_1 - (x_3 - x_0)(x_3 - x_1)\Delta_D^2 f_2$$
$$\Rightarrow a_3(x_3 - x_1)(x_3 - x_2) = \frac{f_3 - f_0}{x_3 - x_0} - \Delta_D^1 f_1 - (x_3 - x_1)\Delta_D^2 f_2$$
$$\Rightarrow a_3(x_3 - x_1)(x_3 - x_2) = \Delta_D^1 f_3 - \Delta_D^1 f_1 - (x_3 - x_1)\Delta_D^2 f_2$$
$$\Rightarrow a_3(x_3 - x_2) = \frac{\Delta_D^1 f_3 - \Delta_D^1 f_1}{x_3 - x_1} - \Delta_D^2 f_2 = \Delta_D^2 f_3 - \Delta_D^2 f_2$$
$$\therefore \quad a_3 = \frac{\Delta_D^2 f_3 - \Delta_D^2 f_2}{x_3 - x_2} = \Delta_D^3 f_3$$

Now continuing in the same manner, at $x = x_n$ we can find that

$$a_n = \frac{\Delta_D^{n-1} f_n - \Delta_D^{n-1} f_{n-1}}{x_n - x_{n-1}} = \Delta_D^n f_n \; .$$

Now substituting the values of $a_0, a_1, a_2, a_3, \ldots, a_n$ in equation (3), we get

$$f(x) \cong \phi(x) = f_0 + (x - x_0)\Delta_D^1 f_1 + (x - x_0)(x - x_1)\Delta_D^2 f_2 + (x - x_0)(x - x_1)(x - x_2)\Delta_D^3 f_3$$
$$+ \ldots\ldots + (x - x_0)(x - x_1)(x - x_2)\ldots\ldots(x - x_{n-1})\Delta_D^n f_n \qquad (4)$$

This is the required general (forward) divided difference interpolating polynomial in terms of x.

5. ***Algorithm for (Forward) Divided Difference Interpolation Formula*:**

Step 1: Read 'The number of data points:', n

Step 2: Set $n = n - 1$

Do $i = 0\ to\ n$

Read x_i, f_i

Set $D^0 f_i = f_i$

End Do

// Construction of Divided Difference Table

Step 3: Do $j = 1\ \ to\ \ n$

Do $i = j\ \ to\ \ n$

Calculate $D^j f_i = (D^{j-1} f_i - D^{j-1} f_{j-1}) / (x_i - x_{j-1})$

End Do (*for* i)

End Do (*for* j)

// End of difference table construction.

Step 4: Read 'The interpolating point:', xp

Step 5: Set $sum = f_0$

Step 6: $prod = 1.0$

Step 7: Do $j = 1\ to\ n$

Calculate $prod = prod \times (xp - x_{j-1})$

Calculate $sum = sum + prod \times D^j f_j$

End Do

// Printing the output value.

Step 8: Print 'The required value = ' , *sum*

Step 9: Stop

6. FORTRAN program based on the *Algorithm in section-5.*

```
C       PROGRAM FOR DIVIDED DIFFERENCE INTERPOLATION
        DIMENSION X(10), F(10), DF(10,10)
        WRITE(*,*) ‘ENTER NUMBER OF DATA POINTS:’
        READ(*,*) N
        N = N-1
        WRITE(*,*) ‘ENTER X VALUES:’
        READ(*,*) (X(J), J = 0, N)
        WRITE(*,*) ‘ENTER F(X) VALUES:’
        READ(*,*) (F(J), J = 0, N)

C       CONSTRUCTION OF DIVIDED DIFFERENCE TABLE
        DO 5 K = 1, N
        DF(0,K) = F(K)
5       CONTINUE
        DO 25 J = 1, N
        DO 25 K = J, N
        DF(J,K) = (DF(J-1,K) - DF(J-1,J-1))/(X(K)-X(J-1))
25      CONTINUE
        WRITE(*,*) ‘ENTER INTERPOLTING POINT:’
        READ(*,*) XP
        SUM = F(0)
        PROD = 1.0
        DO 35 K = 1, N
        PROD = PROD *(XP-X(K-1))
        SUM = SUM + DF(K,K)*PROD
35      CONTINUE
        WRITE(*,55) SUM
55      FORMAT (5X, 'THE REQUIRED VALUE = ', F9.4//)
        END
```

7. *Numerical Illustration with Verification*

⏭ **Problem 7.1:** Establish with verification, an interpolating polynomial that fits the following data points.

x	:	3	2	1	–1
$f(x)$	:	3	12	15	–21

***Solution*:** The difference table for the given data points is presented as follows:

i	x_i	$\Delta_D^0 f_i = f_i$	First Order Difference	Second Order Difference	Third Order Difference
0	3	3			
1	2	12	$\Delta_D^1 f_1 = \frac{12-3}{2-3} = -9$		
2	1	15	$\Delta_D^1 f_2 = \frac{15-3}{1-3} = -6$	$\Delta_D^2 f_2 = \frac{-6-(-9)}{1-2} = -3$	
3	–1	–21	$\Delta_D^1 f_3 = \frac{-21-3}{-1-3} = 6$	$\Delta_D^2 f_3 = \frac{6-(-9)}{-1-2} = -5$	$\Delta_D^3 f_3 = \frac{-5-(-3)}{-1-1} = 1$

The polynomial interpolating formula up to the 3rd order difference is:

$$\phi(x) \cong f(x) = f_0 + (x-x_0)\,\Delta_D^1 f_1 + (x-x_0)(x-x_1)\,\Delta_D^2 f_2 + (x-x_0)(x-x_1)(x-x_2)\,\Delta_D^3 f_3 \qquad (5)$$

Here we have,

$f_0 = 3$, $\Delta_D^1 f_1 = -9$, $\Delta_D^2 f_2 = -3$, and $\Delta_D^3 f_3 = 1$.

Now from equation (5), we have

$$\begin{aligned} f(x) &= 3 + (x-3)(-9) + (x-3)(x-2)\times(-3) + (x-3)(x-2)(x-1)\times(1) \\ &= 3 - 9x + 27 - 3(x^2 - 5x + 6) + (x^2 - 5x + 6)(x-1) \\ &= 3 - 9x + 27 - 3x^2 + 15x - 18 + x^3 - 5x^2 + 6x - x^2 + 5x - 6 \\ &= x^3 + (-3x^2 - 5x^2 - x^2) + (-9x + 15x + 6x + 5x) + (3 + 27 - 18 - 6) \end{aligned}$$

$\therefore\ f(x) = x^3 - 9x^2 + 17x + 6$; which is the required polynomial. **(*Ans.*)**

Verification:

$$\left.\begin{aligned} f(3) &= (3)^3 - 9\times(3)^2 + 17\times(3) + 6 = 27 - 81 + 51 + 6 = 3 \\ f(2) &= (2)^3 - 9\times(2)^2 + 17\times(2) + 6 = 8 - 36 + 34 + 6 = 12 \\ f(1) &= (1)^3 - 9\times(1)^2 + 17\times(1) + 6 = 1 - 9 + 17 + 6 = 15 \\ \text{and } f(-1) &= (-1)^3 - 9\times(-1)^2 + 17\times(-1) + 6 = -1 - 9 - 17 + 6 = -21 \end{aligned}\right\}$$

⏭ **Problem-7.2:** Establish with verification, an interpolating polynomial that fits the following data points.

x	:	0	1	2	5
$f(x)$	:	2	3	12	147

***Solution*:** The difference table for the given data points is presented as follows:

i	x_i	$\Delta_D^0 f_i = f_i$	First Order Difference	Second Order Difference	Third Order Difference
0	0	2			
1	1	3	$\Delta_D^1 f_1 = \frac{3-2}{1-0} = 1$		
2	2	12	$\Delta_D^1 f_2 = \frac{12-2}{2-0} = 5$	$\Delta_D^2 f_2 = \frac{5-1}{2-1} = 4$	
3	5	147	$\Delta_D^1 f_3 = \frac{147-2}{5-0} = 29$	$\Delta_D^2 f_3 = \frac{29-1}{5-1} = 7$	$\Delta_D^3 f_3 = \frac{7-4}{5-2} = 1$

The interpolating polynomial up to the 3rd order difference is:

$$\phi(x) \cong f(x) = f_0 + (x-x_0)\Delta_D^1 f_1 + (x-x_0)(x-x_1)\Delta_D^2 f_2 + (x-x_0)(x-x_1)(x-x_2)\Delta_D^3 f_3 \qquad (6)$$

Here we have,

$$f_0 = 2,\ \Delta_D^1 f_1 = 1,\ \Delta_D^2 f_2 = 4, \text{ and } \Delta_D^3 f_3 = 1.$$

Now from equation (6), we have

$$\begin{aligned} f(x) &= 2 + (x-0)(1) + (x-0)(x-1)\times(4) + (x-0)(x-1)(x-2)\times(1) \\ &= 2 + x + 4(x^2 - x) + (x^2 - x)(x-2) \\ &= 2 + x + 4x^2 - 4x + x^3 - x^2 - 2x^2 + 2x \\ &= x^3 + (4x^2 - x^2 - 2x^2) + (x - 4x + 2x) + 2 \end{aligned}$$

$\therefore f(x) = x^3 + x^2 - x + 2$; which is the required polynomial.

Verification:

$$\left.\begin{aligned} f(0) &= (0)^3 + (0)^2 - 0 + 2 = 2 \\ f(1) &= (1)^3 + (1)^2 - 1 + 2 = 3 \\ f(2) &= (2)^3 + (2)^2 - 2 + 2 = 8 + 4 - 2 + 2 = 12 \\ \text{and } f(5) &= (5)^3 + (5)^2 - 5 + 2 = 125 + 25 - 5 + 2 = 147 \end{aligned}\right\}$$

⏭ **Problem 7.3:** Establish with verification, an interpolating polynomial that fits the following data points.

x	:	0	1	2	4
$f(x)$	:	5	14	41	101

***Solution*:** The divided difference table for the given data points is:

i	x_i	$\Delta_D^0 f_i = f_i$	First Order Difference	Second Order Difference	Third Order Difference
0	0	5			
1	1	14	$\Delta_D^1 f_1 = \frac{14-5}{1-0} = 9$		
2	2	41	$\Delta_D^1 f_2 = \frac{41-5}{2-0} = 18$	$\Delta_D^2 f_2 = \frac{18-9}{2-1} = 9$	
3	4	101	$\Delta_D^1 f_3 = \frac{101-5}{4-0} = 24$	$\Delta_D^2 f_3 = \frac{24-9}{4-1} = 5$	$\Delta_D^2 f_3 = \frac{5-9}{4-2} = -2$

The interpolating polynomial up to the 3rd order difference is:

$$\phi(x) \cong f(x) = f_0 + (x-x_0)\Delta_D^1 f_1 + (x-x_0)(x-x_1)\Delta_D^2 f_2 + (x-x_0)(x-x_1)(x-x_2)\Delta_D^3 f_3 \qquad (7)$$

Here we have,

$f_0 = 5$, $\Delta_D^1 f_1 = 9$, $\Delta_D^2 f_2 = 9$, and $\Delta_D^3 f_3 = -2$.

Now from equation (6), we have

$$\begin{aligned} f(x) &= 5 + (x-0)(9) + (x-0)(x-1)\times 9 + (x-0)(x-1)(x-2)\times(-2) \\ &= 5 + 9x + 9x^2 - 9x - 2(x^3 - 3x^2 + 2) \\ &= 5 + 9x + 9x^2 - 9x - 2x^3 + 6x^2 - 4 \end{aligned}$$

$\therefore \; f(x) = -2x^3 + 15x^2 - 4x + 5$; which is the required polynomial.

Verification:

$$\left.\begin{aligned} f(0) &= -2\times(0)^3 + 15\times(0)^2 - 4(0) + 5 = 5 \\ f(1) &= -2\times1^3 + 15\times1^2 - 4\times1 + 5 = -2 + 15 - 4 + 5 = 14 \\ f(2) &= -2\times2^3 + 15\times2^2 - 4\times2 + 5 = -16 + 60 - 8 + 5 = 41 \\ \text{and } f(4) &= -2\times4^3 + 15\times4^2 - 4\times4 + 5 = -128 + 240 - 16 + 5 = 101 \end{aligned}\right\}$$

8. Conclusion

In this paper, I have introduced a new approach of (forward) divided differences for a given set of unevenly spaced data points in **section-2**. I have also developed a general divided difference polynomial interpolation formula with the help of these divided differences in **section-4**. Three numerical examples are carried out to implement into this new formula and verified by the tabulated points in **section-7**. For computer implementation, an algorithm is designed for this new method in **section-5**. Finally, based on the algorithm, a complete FORTRAN program is provided in **section-6**. In constructing the Newton's divided difference table for a given set of data points, it is very difficult and clumsy process to estimate the divided differences of each order; whereas, in constructing the new divided difference table (**section-4**), the estimation of divided differences is very easy and sophisticated process. Therefore, the new approach of divided difference polynomial interpolation formula is very easy and less laborious method than the existing one.

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