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# ON THE DIOPHANTINE EQUATION $x^2 = y^2 + 3z^2$

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**Abstract**

In an earlier paper, the solutions of the Diophantine equations $a^2 = b^2 + c^2 \pm bc$, which arise in connection with the 60-degree and 120-degree S-related and Z-related triangles, were studied to some extent. This paper considers the Diophantine equation $x^2 = y^2 + 3z^2$, which unravels some new facts about the solutions of the Diophantine equations $a^2 = b^2 + c^2 \pm bc$.

**Keywords:** S-related and Z-related triangles, 60-degree and 120-degree triangles, Diophantine equations

## 1. Introduction

Let T(a, b, c) the the triangle ΔABC with sides a = BC, b = AC, c = AB. The following definition is due to Sastry [1] and Ashbacher [2].

***Definition 1.1* :** Two triangles $T(a, b, c)$ and $T(a', b', c')$ are said to be S-related if

$$S(a) = S(a'),\ S(b) = S(b'),\ S(c) = S(c');$$

and they are said to be Z-related if

$$Z(a) = Z(a'),\ Z(b) = Z(b'),\ Z(c) = Z(c'),$$

where S(n) is the Smarandache function, and Z(n) is the pseudo Smarandache function.

Of particular interest are the 60 degrees and 120 degrees triangles that are Z-related or S-related (in the sense of Definition 1.1). Studying on the this problem, Majumdar [3] has found that such pairs of triangles lead to the Diophantine equations

$$a^2 = b^2 + c^2 \pm bc.$$

In [4], Majumdar gives a partial solution to the above equations. This paper considers the more general Diophantine equation $x^2 = y^2 + 3z^2$. This is done in Section 3. Some preliminary results are given in Section 2. We conclude the paper with some observations and remarks in the final Section 4.

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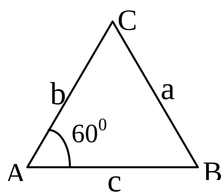
## 2. Some preliminary results

The following two lemmas give respectively the conditions satisfied by the 60 degrees and 120 degrees triangles. The proofs are given in Majumdar [3].

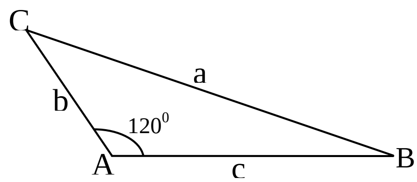
***Lemma 2.1*** **:** Let T(a, b, c) be a triangle with sides a, b and c, whose $\angle A = 60^0$ (as shown in the figure). Then,

$$4a^2 = (2c - b)^2 + 3b^2. \qquad (2.1)$$




***Lemma 2.2*** **:** Let T(a, b, c) be a triangle with sides a, b and c, whose $\angle A = 120^0$ (as shown in the figure). Then,

$$4a^2 = (2c + b)^2 + 3b^2. \qquad (2.2)$$




Clearly, a = b = c is a solution of the Diophantine equation (2.1), called its trivial solution. For non-trivial solutions, as has already been observed in Majumdar [3], when $\angle A = 60^0$, $\min\{b, c\} \le a \le \max\{b, c\}$; and if $\angle A = 120^0$, then a is the largest side of the triangle ABC. Thus, if $(a_0, b_0, c_0)$ is a solution of the Diophantine equation $4a^2 = (2c - b)^2 + 3b^2$, then, without loss of generality, we may assume that

$$c_0 < a_0 < b_0. \qquad (2.3)$$

The lemma below shows that the Diophantine equation $4a^2 = (2c - b)^2 + 3b^2$ possesses (independent) solutions in pair, namely, $(a_0, b_0, c_0)$ and $(a_0, b_0, b_0 - c_0)$. Note that, by symmetry, $(a_0, c_0, b_0)$ and $(a_0, b_0 - c_0, b_0)$ are also solutions of the Diophantine equation.

***Lemma 2.3*** **:** If $(a_0, b_0, c_0)$ is a solution of the Diophantine equation

$$4a^2 = (2c - b)^2 + 3b^2,$$

then $(a_0, b_0, b_0 - c_0)$ is also a solution of the Diophantine equation (2.2).

Next, we confine our attention to the Diophantine equation

$$4a^2 = (2c + b)^2 + 3b^2.$$

The following lemma, due to Majumdar [3], gives the relationship between the solutions of the Diophantine equations (2.1) and (2.2).

***Lemma 2.4* :** If $(a_0, b_0, c_0)$ is a solution of the Diophantine equation $4a^2 = (2c - b)^2 + 3b^2$, then $(a_0, b_0 - c_0, c_0)$ is a solution of the Diophantine equation $4a^2 = (2c + b)^2 + 3b^2$.

Note that, the Diophantine equations (2.1) and (2.2) are equivalent to the Diophantine equations

$$a^2 = b^2 + c^2 \pm bc,$$

and by virtue of Lemma 2.4, it is sufficient to concentrate on the Diophantine equation

$$a^2 = b^2 + c^2 - bc \qquad (2.4)$$

only.

In Majumdar [3], it has been pointed out that, in certain cases, the Diophantine equation (2.1) has more than two independent solutions. For better understanding of the situation, we consider the more general Diophantine equation

$$x^2 = y^2 + 3z^2. \qquad (2.5)$$

Note that the Diophantine equation (2.1) is a particular case of the above equation with

$$x = 2a,\ y = 2b - c,\ z = c. \qquad (2.6)$$

It may be mentioned here that, when $b < c$, we may still consider the above equation (2.5) with the roles of b and c interchanged.

In the next section, we study the Diophantine equation (2.5) more closely.

## 3. The Diophantine Equation $x^2 = y^2 + 3z^2$

In this section, we consider the Diophantine equation

$$x^2 = y^2 + 3z^2. \qquad (3.1)$$

For example, when $x = 14 = 2\times7$, there are three (independent) solutions of (3.1), as shown below :

$$14^2 = 13^2 + 3.3^2 = 11^2 + 3.5^2 = 2^2 + 3.8^2, \qquad (1)$$

which in turn shows that, corresponding to $x = 7$, there is only one solution.

The relationship between the solutions of the Diophantine equations (3.1) and (2.4) is expressed in the lemma below.

***Lemma 3.1* :** Let $(x_0, y_0, z_0)$ be a solution of the Diophantine equation $x^2 = y^2 + 3z^2$. Then, $(a_0, b_0, c_0)$ is a solution of the Diophantine equation $a^2 = b^2 + c^2 - bc$, where

$$a_0 = \frac{x_0}{2},\ b_0 = \frac{1}{2}(y_0 + z_0),\ c_0 = z_0.$$

***Proof.*** Follows from (2.6), noting that $y_0 = 2b_0 - c_0 > 0$ when $b_0 > c_0$.

It may be mentioned here that Lemma 3.1 may be applied when $c_0 > b_0$, by interchanging the roles of $b_0$ and $c_0$.

Now, given a solution $(x_0, y_0, z_0)$ of the Diophantine equation (3.1), its solution when $x = x_0^2$ may be found as follows.

***Lemma 3.2* :** Let $(x_0, y_0, z_0)$ be a solution of the Diophantine equation $x^2 = y^2 + 3z^2$. Then, $(x_0^2, |y_0^2 - 3z_0^2|, 2y_0 z_0)$ is also a solution of it.

***Proof.*** Writing $(y^2 + 3z^2)^2$ in the form below :

$$(y^2 + 3z^2)^2 = (y^2 - 3z^2)^2 + 12\, y^2 z^2 = (y^2 - 3z^2)^2 + 3(2yz)^2,$$

the result follows immediately.

If $(x_0, y_0, z_0)$ is a solution of $x^2 = y^2 + 3z^2$, then obviously $(x_0^2, x_0 y_0, x_0 z_0)$ is also its one solution; Lemma 3.2 gives another independent solution of it. For example, starting with the solution, given in (1), we get, by Lemma 3.2, three solutions when $x = 14^2$, namely,

$$(14^2, 142, 78), (14^2, 46, 110), (14^2, 188, 32),$$

the last one giving the solution $(7^2, 47, 8)$. Thus, corresponding to $x = 14^2$, there are, in total, six solutions of the Diophantine equation (3.1)

***Proposition 3.1* :** Let $(a_0, b_0, c_0)$ be a solution of the Diophantine equation $a^2 = b^2 + c^2 - bc$. Then, $(a_0^2, b_0^2 - c_0^2, c_0(2b_0 - c_0))$ is also a solution of it.

***Proof.*** By Lemma 3.1,

$$a_0 = \frac{x_0}{2},\ b_0 = \frac{1}{2}(y_0 + z_0),\ c_0 = z_0.$$

Now, by Lemma 3.2, if $(a_1, b_1, c_1)$ is a solution of (2.1) corresponding to $a = a_1 = a_0^2$, then

$$a_1 = \frac{x_0^2}{2},\ 2b_1 - c_1 = \left|y_0^2 - 3z_0^2\right|,\ c_1 = 2y_0 z_0.$$

Therefore, if $y_0^2 - 3z_0^2 > 0$, then

$$2b_1 = 2y_0 z_0 + (y_0^2 - 3z_0^2) = (z_0 + y_0)^2 - 4z_0^2 = 4(b_0^2 - c_0^2),$$

so that

$$b_1 = 2(b_0^2 - c_0^2);$$

also,

$$a_1 = 2a_0^2,\ c_1 = 2c_0(2b_0 - c_0).$$

Now, disregarding the common factor 2, the result follows.

If the solution $(x_0, y_0, z_0)$ of the Diophantine equation is known, then using Lemma 3.1, we may find the corresponding solution of the Diophantine equation (2.1). And if a solution $(a_0, b_0, c_0)$ of the Diophantine equation (2.1) is known, Proposition 3.1 may be employed to find the solution corresponding to $a = a_0^2$. For example, from the first two solutions given in (1), we get (by Lemma 3.1)

$$a_0 = 7,\ b_0 = 8,\ c_0 = 3, \tag{2}$$

$$a_0 = 7,\ b_0 = 8,\ c_0 = 5, \tag{3}$$

while the third one in (1) gives, after interchanging the roles of $b_0$ and $c_0$,

$$a_0 = 7,\ b_0 = 3,\ c_0 = 8.$$

Now, applying Proposition 3.1 to the solution (2), we get

$$a_1 = 7^2,\ b_1 = 55,\ c_1 = 39, \tag{4}$$

while (3) gives

$$a_1 = 7^2,\ b_1 = 39,\ c_1 = 55,$$

which is just the solution (4) with the roles of $b_1$ and $c_1$ interchanged.

Thus, from the three solutions of the Diophantine equation (3.1), given in (1), we get only one (distinct) solution of the Diophantine equation (2.1), by applying Proposition 3.1.

The lemma below considers the case when two independent solutions of the Diophantine equation (3.1) are known.

***Lemma 3.3* :** Let (X, Y, Z) and (A, B, C) be two independent solutions of the Diophantine equation $x^2 = y^2 + 3z^2$. Then, $(XA, |BY - 3CZ|, BZ + CY)$ and $(XA, BY + 3CZ, |BZ - CY|)$ are also its solutions.

***Proof.*** Note that $X^2A^2 = (Y^2 + 3Z^2)(B^2 + 3C^2)$ can be expressed in two ways as follows :

$$(Y^2 + 3Z^2)(B^2 + 3C^2) = (BY - 3CZ)^2 + 3(BZ + CY)^2 = (BY + 3CZ)^2 + 3(BZ - CY)^2.$$

Thus, we get the desired result.

If two independent solutions of $x^2 = y^2 + 3z^2$ are known, Lemma 3.3 finds two more. Since

$$26^2 = 23^2 + 3.7^2 = 22^2 + 3.8^2 = 1^2 + 3.15^2, \qquad (5)$$

from (1) and (5), by Lemma 3.2, we get the following six independent solutions :

$$(14\times 26)^2 = 236^2 + 3.160^2 = 362^2 + 3.22^2$$

$$= 148^2 + 3.192^2 = 358^2 + 3.38^2$$

$$= 122^2 + 3.198^2 = 214^2 + 3.170^2.$$

Thus, corresponding to $x = 7\times 13$, there are only two independent solutions of (3.1), namely,

$$(7\times 13)^2 = 59^2 + 3.40^2 = 37^2 + 3.48^2.$$

***Proposition 3.2 :*** Let $(a_0, b_0, c_0)$ and $(a_1, b_1, c_1)$ be two independent solutions of the Diophantine equation $a^2 = b^2 + c^2 - bc$. Then, $(a_0 a_1, b_0 b_1 - c_0 c_1, c_0(b_1 - c_1) + b_0 c_1)$ and $(a_0 a_1, b_0(b_1 - c_1) + c_0 c_1, b_1 c_0 - b_0 c_1)$ are also its solutions, where in the latter case, we must have $b_1 c_0 - b_0 c_1 > 0$; if $b_1 c_0 - b_0 c_1 < 0$, interchange the roles of the first and the second solutions.

***Proof.*** By Lemma 3.1,

$$a_0 = \frac{X}{2},\ b_0 = \frac{1}{2}(Y+Z),\ c_0 = Z,$$

$$a_1 = \frac{A}{2},\ b_1 = \frac{1}{2}(B+C),\ c_1 = C.$$

Now, by Lemma 3.2,

$$a_2 = \frac{XA}{2},\ b_2 = \frac{1}{2}\left[(BY-3CZ)+(BZ+CY)\right],\ c_2 = BZ+CY,$$

$$a_3 = \frac{XA}{2},\ b_3 = \frac{1}{2}\left[(BY+3CZ)+(BZ-CY)\right],\ c_3 = BZ-CY.$$

We now simplify as follows :

$$a_2 = 2a_0 a_1,\ c_2 = (2b_1 - c_1)c_0 + (2b_0 - c_0)c_1 = 2(b_0 c_1 + b_1 c_0 - c_0 c_1),$$

$$BY - 3CZ = (2b_1 - c_1)(2b_0 - c_0) - 3c_1 c_0 = 2(2b_0 b_1 - b_0 c_1 - b_1 c_0 - c_0 c_1),$$

so that

$$b_2 = (2b_0 b_1 - b_0 c_1 - b_1 c_0 - c_0 c_1) + (b_0 c_1 + b_1 c_0 - c_0 c_1) = 2(b_0 b_1 - c_0 c_1).$$

Again,

$$a_3 = 2a_0 a_1,\ c_3 = (2b_1 - c_1)c_0 - (2b_0 - c_0)c_1 = 2(b_1 c_0 - b_0 c_1),$$

$$BY + 3CZ = (2b_1 - c_1)(2b_0 - c_0) + 3c_1 c_0 = 2(2b_0 b_1 + 2c_0 c_1 - b_0 c_1 - b_1 c_0),$$

so that

$$b_3 = (2b_0 b_1 + 2c_0 c_1 - b_0 c_1 - b_1 c_0) + (b_1 c_0 - b_0 c_1) = 2(b_0 b_1 + c_0 c_1 - b_0 c_1).$$

Now, ignoring the common factor 2, we get the desired result.

If $(a_0, b_0, c_0)$ and $(a_1, b_1, c_1)$ are two independent solutions of the Diophantine equation $a^2 = b^2 + c^2 - bc$, then obviously $(a_0 a_1, b_0 a_1, c_0 a_1)$ and $(a_0 a_1, a_0 b_1, a_0 c_1)$ are its two independent solutions. By virtue of Proposition 3.2, we get two more independent solutions corresponding to $a = a_0 a_1$. Since

$$a_0 = 13, b_0 = 15, c_0 = 7, \qquad (6)$$

$$a_0 = 13, b_0 = 15, c_0 = 8, \qquad (7)$$

are solutions of the Diophantine equation (2.1), the two solutions (3) and (6) together gives, by Proposition 3.2, the two independent solutions

$$a_2 = 7\times13, b_2 = 85, c_2 = 96,$$

$$a_2 = 7\times13, b_2 = 99, c_2 = 19.$$

Thus, the four independent solutions of the Diophantine equation (2.1), obtained by the application of Proposition 3.2, are

$$a_2 = 7\times13, b_2 = 96, c_2 = 85.$$

$$a_2 = 7\times13, b_2 = 96, c_2 = 11.$$

$$a_2 = 7\times13, b_2 = 99, c_2 = 80.$$

$$a_2 = 7\times13, b_2 = 96, c_2 = 19.$$

This explains why the Diophantine equation (2.1) possesses eight independent solutions corresponding to $a = 7\times13$.

## 4. Remarks

Corresponding to $a = 7$, the two independent solutions of the Diophantine equation (2.1) are given in (2) and (3), and its solutions corresponding to $a = 13$ are given in (6) and (7). In Section 3 above, we found the four independent solutions of (2.1) when $a = 7\times13$, by considering the solutions (3) and (6). If we want to apply the second part of Proposition 3.2, we see that, with the two solutions (2) and (6) (in this order), $c_3 = -11 < 0$. Thus, we rewrite them as

$a_0 = 13$, $b_0 = 15$, $c_0 = 7$,

$a_1 = 7$, $b_1 = 8$, $c_1 = 3$,

and then, by Proposition 3.2,

$a_2 = 7\times 13$, $b_2 = 99$, $c_2 = 80$,

$a_2 = 7\times 13$, $b_2 = 96$, $c_2 = 11$.

It can be seen easily that the two solutions obtained from (3) and (7) are not distinct from the two solutions obtained above.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